

Compact operators on Hilbert right modules

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Abstract

We generalize some results on compact operators on Hilbert spaces to "compact" operators on some Hilbert right W^* -modules. We present in this frame the Schatten decomposition of the compact operators, the trace, the Banach \mathcal{L}^p -spaces and their duality, the Hilbert-Schmitt operators, and the integral operators as an example of Hilbert-Schmitt operators.

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0 Notation and terminology

In general we use the notation and terminology of [C]. In the sequel we give a list of such notation and terminology from [C] used in this paper.

1. \mathbb{K} denotes the field of real numbers \mathbb{R} or the field of complex numbers \mathbb{C} . The whole theory is developed in parallel for the real and complex case, but the proofs coincide. \mathbb{Z} denotes the set of integers, \mathbb{N} denotes the set of natural numbers ($0 \notin \mathbb{N}$) and we put for every $n \in \mathbb{N}$,

$$\mathbb{N}_n := \{ k \in \mathbb{N} \mid k \leq n \}.$$

An initial segment of \mathbb{N} is a subset N of \mathbb{N} such that given $m \in \mathbb{N}$ and $n \in N$, with $m < n$, then $m \in N$. \mathbb{R}_+ denotes the set of positive real numbers ($0 \in \mathbb{R}_+$).

2. If A is a set then id_A denotes the identity map of A .
3. If E is a Banach space then $E^\#$ denotes the unit ball of E :

$$E^\# := \{ x \in E \mid \|x\| \leq 1 \}.$$

If T is a compact space then $\mathcal{C}(T, E)$ denotes the Banach space of continuous maps $T \rightarrow E$ (endowed with the supremum norm). We put $\mathcal{C}(T) := \mathcal{C}(T, \mathbb{K})$.

4. Let E be a C^* -algebra. We denote by E_+ the set of positive elements of E and put $E_+^\# := E_+ \cap E^\#$. If E is unital then 1_E denotes its unit. For $x \in E$, $\sigma(x)$ denotes the spectrum of x .

5. If I is a set, then $l^2(I)$ denotes the Hilbert space of square summable families in \mathbb{K} indexed by I , $\mathcal{L}(l^2(I))$ the W^* -algebra of operators

$$l^2(I) \rightarrow l^2(I),$$

and $\mathcal{K}(l^2(I))$ the C^* -subalgebra of $\mathcal{L}(l^2(I))$ of compact operators.

6. δ_{ij} denotes Kronecker's symbol:

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

7. Let E be a C^* -algebra and H a Hilbert right E -module. We denote by $\mathcal{L}(H)$ the Banach space of operators $H \rightarrow H$, by $\mathcal{L}_E(H)$ its Banach subspace of adjointable operators, which is a C^* -algebra, and by $\mathcal{K}_E(H)$ the C^* -subalgebra of $\mathcal{L}_E(H)$ of "compact" operators. For all $\xi, \eta \in H$ we denote by $\langle \xi \mid \eta \rangle$ their scalar product and put

$$\xi \langle \cdot \mid \eta \rangle : H \longrightarrow H, \quad \zeta \longmapsto \xi \langle \zeta \mid \eta \rangle.$$

Throughout this paper we denote by T a compact hyperstonian space ([C] Definition 1.7.2.12), by $E := \mathcal{C}(T)$ the C^* -algebra of continuous scalar valued functions on T (by [C] Theorem 4.4.4.22 c \Rightarrow a, E is a W^* -algebra), by K a selfdual Hilbert right E -module, by $(p_\iota)_{\iota \in I}$ a family of orthogonal projection of E such that K is isomorphic to $\bigoplus_{\iota \in I}^W p_\iota E$ ([C] Proposition 5.6.4.10 a)), and put $H := \bigoplus_{\iota \in I} p_\iota E$ (by [C] Proposition 5.6.4.1 c), H is a Hilbert right E -module)

1 The C*-algebra $\mathcal{K}_E(H)$

Definition 1.1 We define ψ and for every $t \in T$, ψ_t and φ_t by

$$\begin{aligned}\psi : l^2(I) &\longrightarrow H, \quad \zeta \longmapsto (\zeta_i p_i)_{i \in I} \\ \psi_t : H &\longrightarrow l^2(I), \quad \xi \longmapsto (\xi_i(t))_{i \in I}, \\ \varphi_t : \mathcal{L}_E(H) &\longrightarrow \mathcal{L}(l^2(I)), \quad u \longmapsto \psi_t \circ u \circ \psi.\end{aligned}$$

Proposition 1.2 For every $\xi \in H$ the map

$$T \longrightarrow l^2(I), \quad t \longmapsto \psi_t \xi$$

is continuous.

Let $\varepsilon > 0$. There is a finite subset J of I such that

$$\sum_{i \in I \setminus J} |\xi_i(t)|^2 < \varepsilon$$

for all $t \in T$. For $t, t' \in T$,

$$\begin{aligned}\|\psi_t \xi - \psi_{t'} \xi\|^2 &= \sum_{i \in I} |\xi_i(t) - \xi_i(t')|^2 \leq \\ &\leq \sum_{i \in J} |\xi_i(t) - \xi_i(t')|^2 + 2 \sum_{i \in I \setminus J} |\xi_i(t)|^2 + 2 \sum_{i \in I \setminus J} |\xi_i(t')|^2 \leq \\ &\leq \sum_{i \in J} |\xi_i(t) - \xi_i(t')|^2 + 4\varepsilon,\end{aligned}$$

and this implies the assertion. ■

Proposition 1.3 Let $t \in T$.

$$a) \quad \psi_t \circ \psi \circ \psi_t = \psi_t.$$

b) For $\xi, \eta \in H$ and $\zeta \in l^2(I)$,

$$\langle \psi_t \xi | \psi_t \eta \rangle = (\langle \xi | \eta \rangle)(t),$$

$$\langle \psi_t \xi | \zeta \rangle = \langle \psi_t \xi | \psi_t \psi \zeta \rangle = (\langle \xi | \psi \zeta \rangle)(t).$$

c) For every $u \in \mathcal{L}_E(H)$,

$$\psi_t \circ u \circ \psi \circ \psi_t = \psi_t \circ u.$$

d) For $u, v \in \mathcal{L}_E(H)$,

$$\varphi_t(uv) = (\varphi_t u)(\varphi_t v).$$

e) For every $u \in \mathcal{L}_E(H)$,

$$\varphi_t u^* = (\varphi_t u)^*.$$

f) For $\xi, \eta \in H$,

$$\varphi_t(\xi \langle \cdot | \eta \rangle) = (\psi_t \xi) \langle \cdot | \psi_t \eta \rangle.$$

a) and b) are easy to see.

c) For $\xi \in H$, by a), $\psi_t(\xi - \psi \psi_t \xi) = 0$. Let $\varepsilon > 0$. By Proposition 1.2 there is a neighborhood U of t such that $\|\psi_{t'}(\xi - \psi \psi_{t'} \xi)\| < \varepsilon$ for every $t' \in U$. Let $x \in E_+^\#$ with $x(t) = 1$ and $x = 0$ on $T \setminus U$. Then $\|(\xi - \psi \psi_t \xi)x\| < \varepsilon$ and

$$\|(u(\xi - \psi \psi_t \xi))x\| = \|u((\xi - \psi \psi_t \xi)x)\| \leq \varepsilon \|u\|,$$

$$\|\psi_t(u(\xi - \psi \psi_t \xi))\| = \|\psi_t((u(\xi - \psi \psi_t \xi))x)\| \leq \varepsilon \|u\|.$$

Since ε is arbitrary,

$$\psi_t u \xi = \psi_t u \psi \psi_t \xi, \quad \psi_t \circ u = \psi_t \circ u \circ \psi \circ \psi_t.$$

d) For $\zeta \in l^2(I)$, by c),

$$(\varphi_t u)(\varphi_t v)\zeta = \psi_t u \psi \psi_t v \psi \zeta = \psi_t u v \psi \zeta = (\varphi_t(uv))\zeta,$$

$$(\varphi_t u)(\varphi_t v) = \varphi_t(uv).$$

e) For $\xi, \eta \in l^2(I)$, by b),

$$\begin{aligned}\langle \xi | (\varphi_t u)^* \eta \rangle &= \langle (\varphi_t u) \xi | \eta \rangle = \langle \psi_t u \psi \xi | \psi_t \psi \eta \rangle = (\langle u \psi \xi | \psi \eta \rangle)(t) = \\ &= (\langle \psi \xi | u^* \psi \eta \rangle)(t) = \langle \psi_t \psi \xi | \psi_t u^* \psi \eta \rangle = \langle \xi | (\varphi_t u^*) \eta \rangle, \\ &(\varphi_t u)^* = \varphi_t u^*.\end{aligned}$$

f) For $\zeta \in l^2(I)$, by b),

$$\begin{aligned}\varphi_t(\xi \langle \cdot | \eta \rangle) \zeta &= \psi_t((\xi \langle \cdot | \eta \rangle) \psi \zeta) = \psi_t(\xi \langle \psi \zeta | \eta \rangle) = (\psi_t \xi)(\langle \psi \zeta | \eta \rangle)(t) = \\ &= (\psi_t \xi) \langle \psi_t \psi \zeta | \psi_t \eta \rangle = (\psi_t \xi) \langle \zeta | \psi_t \eta \rangle = ((\psi_t \xi) \langle \cdot | \psi_t \eta \rangle) \zeta, \\ &\varphi_t(\xi \langle \cdot | \eta \rangle) = (\psi_t \xi) \langle \cdot | \psi_t \eta \rangle.\end{aligned}$$

■

Corollary 1.4

a) The map

$$\mathcal{L}_E(H) \longrightarrow \prod_{t \in T} \mathcal{L}(l^2(I)), \quad u \longmapsto (\varphi_t u)_{t \in T}$$

is an injective C^* -homomorphism.

b) $u \in \mathcal{L}_E(H)$ is positive iff $\varphi_t u$ is positive for all $t \in T$.

a) By Proposition 1.3 d),e), the map

$$\mathcal{L}_E(H) \longrightarrow \prod_{t \in T} \mathcal{L}(l^2(I)), \quad u \longmapsto (\varphi_t u)_{t \in T}$$

is a C^* -homomorphism. Let $u \in \mathcal{L}_E(H)$ such that $\varphi_t u = 0$ for all $t \in T$. For $\xi \in H$ and $t \in T$, by Proposition 1.3 c),

$$\psi_t u \xi = \psi_t u \psi \psi_t \xi = (\varphi_t u) \psi_t \xi = 0, \quad u \xi = 0, \quad u = 0,$$

so the above map is injective.

b) follows from a).

■

Proposition 1.5

a) For every $u \in \mathcal{K}_E(H)$ the map

$$\bar{u} : T \longrightarrow \mathcal{K}(l^2(I)) , \quad t \longmapsto \varphi_t u$$

is continuous.

b) The map

$$\mathcal{K}_E(H) \longrightarrow \mathcal{C}(T, \mathcal{K}(l^2(I))) , \quad u \longmapsto \bar{u}$$

is an injective C^* -homomorphism.

a) Let $\xi, \eta \in H$ and $t, t' \in T$. By Proposition 1.3 f),

$$\begin{aligned} \varphi_t(\xi \langle \cdot | \eta \rangle) - \varphi_{t'}(\xi \langle \cdot | \eta \rangle) &= (\psi_t \xi) \langle \cdot | \psi_t \eta \rangle - (\psi_{t'} \xi) \langle \cdot | \psi_{t'} \eta \rangle = \\ &= (\psi_t \xi) \langle \cdot | \psi_t \eta \rangle - (\psi_t \xi) \langle \cdot | \psi_{t'} \eta \rangle + (\psi_t \xi) \langle \cdot | \psi_{t'} \eta \rangle - (\psi_{t'} \xi) \langle \cdot | \psi_{t'} \eta \rangle = \\ &= (\psi_t \xi) \langle \cdot | \psi_t \eta - \psi_{t'} \eta \rangle + (\psi_t \xi - \psi_{t'} \xi) \langle \cdot | \psi_{t'} \eta \rangle , \end{aligned}$$

so by [C] Proposition 5.6.5.2 a),

$$\begin{aligned} \|\varphi_t(\xi \langle \cdot | \eta \rangle) - \varphi_{t'}(\xi \langle \cdot | \eta \rangle)\| &\leq \\ &\leq \|(\psi_t \xi) \langle \cdot | \psi_t \eta - \psi_{t'} \eta \rangle\| + \|(\psi_t \xi - \psi_{t'} \xi) \langle \cdot | \psi_{t'} \eta \rangle\| \leq \\ &\leq \|\psi_t \xi\| \|\psi_t \eta - \psi_{t'} \eta\| + \|\psi_t \xi - \psi_{t'} \xi\| \|\psi_{t'} \eta\| \leq \\ &\leq \|\xi\| \|\psi_t \eta - \psi_{t'} \eta\| + \|\psi_t \xi - \psi_{t'} \xi\| \|\eta\| . \end{aligned}$$

Thus by Proposition 1.2, the map

$$T \longrightarrow \mathcal{K}(l^2(I)) , \quad t \longmapsto \varphi_t(\xi \langle \cdot | \eta \rangle)$$

is continuous.

The assertion follows now from the definition of $\mathcal{K}_E(H)$ ([C] Definition 5.6.5.3).

b) follows from a) and Corollary 1.4 a). ■

2 The C*-algebra $\mathcal{C}(T, \mathcal{K}(l^2(I)))$

Proposition 2.1 *Let $u \in \mathcal{C}(T, \mathcal{K}(l^2(I)))$ and $n \in \mathbb{N}$.*

a) *The map $\theta_n(u)$ defined by*

$$\theta_n(u) : T \longrightarrow \mathbb{R}_+, \quad t \longmapsto \theta_n(u(t))$$

(with the notation of [C] Definition 6.1.2.1) is continuous.

b) $\theta_n(u) = \theta_n(u^*) = \theta_n(|u|)$.

c) *If u is positive and f is a continuous increasing function on \mathbb{R}_+ with $f(0) = 0$ then $\theta_n(f(u)) = f(\theta_n(u))$.*

a) follows from [C] Corollary 6.1.2.8.

b) follows from [C] Theorem 6.1.3.1 b).

c) follows from [C] Corollary 6.1.2.16. ■

Proposition 2.2 *If $\xi, \eta \in H$ then*

$$\theta_1(\xi \langle \cdot | \eta \rangle) : T \longrightarrow \mathbb{R}_+, \quad t \longmapsto \|\psi_t \xi\| \|\psi_t \eta\|$$

and $\theta_n(\xi \langle \cdot | \eta \rangle) = 0$ for all $n \in \mathbb{N} \setminus \{1\}$.

For $n \in \mathbb{N}$ and $t \in T$, by Proposition 1.3 f), Proposition 1.5 a), and [C] Proposition 6.1.2.3,

$$\begin{aligned} (\theta_n(\xi \langle \cdot | \eta \rangle))(t) &= \theta_n(\varphi_t(\xi \langle \cdot | \eta \rangle)) = \\ &= \theta_n((\psi_t \xi) \langle \cdot | \psi_t \eta \rangle) = \begin{cases} \|\psi_t \xi\| \|\psi_t \eta\| & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}. \end{aligned} \quad \blacksquare$$

Definition 2.3 *We put for every $\xi \in K$ and $t \in T$,*

$$\boldsymbol{\xi}(t) := (\xi_\iota(t))_{\iota \in I} \in l^2(I).$$

We put for every $u \in \mathcal{C}(T, \mathcal{K}(l^2(I)))$ and $n \in \mathbb{N}$

$$\mathbf{U}_n(\mathbf{u}) := \{t \in T \mid \theta_n(u(t)) \neq 0\},$$

$$\mathbf{e}_n(\mathbf{u}) : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} 1 & \text{if } t \in \overline{U_n(u)} \\ 0 & \text{if } t \in T \setminus \overline{U_n(u)} \end{cases}.$$

A sequence $(\xi_n)_{n \in \mathbb{N}}$ in K is called **u-orthonormal** if for all $m, n \in \mathbb{N}$, $m \leq n$,

$$\langle \xi_m \mid \xi_n \rangle = \delta_{m,n} e_n(u)$$

and the map

$$U_n(u) \longrightarrow l^2(I), \quad t \longmapsto \xi_n(t)$$

is continuous. We extend the above notation and terminology to $u \in \mathcal{K}_E(H)$ by using Proposition 1.5 a).

If $\xi \in H$ then $\xi(t) = \psi_t \xi$ for all $t \in T$.

Proposition 2.4 *Let $u \in \mathcal{C}(T, \mathcal{K}(l^2(I)))$ and let $(\xi_n)_{n \in \mathbb{N}}$ be a u-orthonormal sequence in K .*

a) $U_n(u)$ is the union of a sequence of pairwise disjoint clopen sets of T for every $n \in \mathbb{N}$.

b) $\xi_n \langle \cdot \mid \xi_n \rangle$ is an orthogonal projection of $\mathcal{K}_E(K)$ for every $n \in \mathbb{N}$ and

$$(\xi_m \langle \cdot \mid \xi_m \rangle)(\xi_n \langle \cdot \mid \xi_n \rangle) = 0$$

for all distinct $m, n \in \mathbb{N}$.

a) If we denote for every $k \in \mathbb{Z}$ by U_k the closure of the interior of the set

$$\{t \in T \mid 2^k \leq \theta_n(u(t)) < 2^{k+1}\}$$

then $(U_k)_{k \in \mathbf{Z}}$ is a countable set of pairwise disjoint clopen sets of T the union of which is T .

b) For all $m, n \in \mathbb{N}$, $m \leq n$,

$$(\xi_m \langle \cdot | \xi_m \rangle)(\xi_n \langle \cdot | \xi_n \rangle) = (\xi_m \langle \xi_n | \xi_m \rangle) \langle \cdot | \xi_n \rangle = \delta_{m,n} \xi_m \langle \cdot | \xi_n \rangle. \quad \blacksquare$$

Proposition 2.5 *Let u be a selfadjoint element of $\mathcal{C}(T, \mathcal{K}(l^2(I)))$.*

a) *For every $t \in T$ there is a representation*

$$u(t) = \sum_{\alpha \in \sigma(u(t))} \alpha \pi_{t,\alpha},$$

where for every $\alpha \in \sigma(u(t))$, $\pi_{t,\alpha}$ is the orthogonal projection of $l^2(I)$ onto $\text{Ker}(\alpha 1 - u(t))$ (here $1 = \text{id}_{l^2(I)}$) and $\pi_{t,\alpha} \pi_{t,\beta} = 0$ for all distinct $\alpha, \beta \in \sigma(u(t))$.

b) *Let $t \in T$, $\alpha \in \sigma(u(t))$, $\alpha \neq 0$, $\varepsilon > 0$, and U a neighborhood of α such that $\sigma(u(t)) \cap \bar{U} = \{\alpha\}$ and $|\alpha - \beta| \leq \frac{|\alpha|\varepsilon}{2}$ for all $\beta \in U$. Then there is a neighborhood V of t such that for every $t' \in V$,*

$$\left\| \sum_{\beta \in \sigma(u(t')) \cap U} \beta \pi_{t',\beta} - \alpha \pi_{t,\alpha} \right\| < \varepsilon, \quad \left\| \sum_{\beta \in \sigma(u(t')) \cap U} \pi_{t',\beta} - \pi_{t,\alpha} \right\| < \varepsilon.$$

a) follows from [C] Theorem 5.5.6.1 a \Rightarrow c&e.

b) Let U' be a neighborhood of $\sigma(u(t)) \setminus \{\alpha\}$ such that $\bar{U} \cap \bar{U}' = \emptyset$. By [C] Corollary 2.2.5.2, there is a neighborhood W of t such that $\sigma(u(t')) \subset U \cup U'$ for all $t' \in W$. Let $f \in \mathcal{C}(\mathbb{K})_+$, $0 \leq f \leq 1$, $f = 1$ on \bar{U} , and $f = 0$ on \bar{U}' . By [C] Proposition 4.1.3.20, the map

$$T \longrightarrow \mathcal{K}(l^2(I)), \quad t' \longmapsto f(u(t'))$$

is continuous. Thus there is a neighborhood V of t , $V \subset W$, such that for every $t' \in V$,

$$\|f(u(t')) - f(u(t))\| < \inf \left\{ \varepsilon, \frac{|\alpha|\varepsilon}{2} \right\}.$$

By [C] Theorem 5.5.6.1 $a \Rightarrow f$,

$$f(u(t)) = \alpha \pi_{t,\alpha}, \quad f(u(t')) = \sum_{\beta \in \sigma(u(t')) \cap U} \beta \pi_{t',\beta}.$$

It follows

$$\begin{aligned} \left\| \sum_{\beta \in \sigma(u(t')) \cap U} \beta \pi_{t',\beta} - \alpha \pi_{t,\alpha} \right\| &= \|f(u(t')) - f(u(t))\| < \inf \left\{ \varepsilon, \frac{|\alpha|\varepsilon}{2} \right\}, \\ \left\| \sum_{\beta \in \sigma(u(t')) \cap U} \pi_{t',\beta} - \pi_{t,\alpha} \right\| &= \frac{1}{|\alpha|} \left\| \sum_{\beta \in \sigma(u(t')) \cap U} \alpha \pi_{t',\beta} - \alpha \pi_{t,\alpha} \right\| \leq \\ &\leq \frac{1}{|\alpha|} \left\| \sum_{\beta \in \sigma(u(t')) \cap U} (\alpha - \beta) \pi_{t',\beta} \right\| + \frac{1}{|\alpha|} \left\| \sum_{\beta \in \sigma(u(t')) \cap U} \beta \pi_{t',\beta} - \alpha \pi_{t,\alpha} \right\| \leq \\ &\leq \frac{|\alpha - \beta|}{|\alpha|} + \frac{1}{|\alpha|} \frac{|\alpha|\varepsilon}{2} \leq \varepsilon. \end{aligned} \quad \blacksquare$$

Lemma 2.6 *Let $\eta : T \longrightarrow l^2(I)$ be a map such that the map*

$$T \longrightarrow \mathcal{K}(l^2(I)), \quad t \longmapsto \eta(t) \langle \cdot | \eta(t) \rangle$$

is continuous. Let $t_0 \in T$ with $\eta(t_0) \neq 0$ and put

$$U := \{ t \in T \mid \langle \eta(t_0) | \eta(t) \rangle \neq 0 \},$$

$$\xi : U \longrightarrow l^2(I), \quad t \longmapsto \frac{\langle \eta(t_0) | \eta(t) \rangle}{|\langle \eta(t_0) | \eta(t) \rangle|} \eta(t).$$

Then U is an open neighborhood of t_0 , ξ is continuous, $\xi(t_0) = \eta(t_0)$, and

$$\xi(t) \langle \cdot | \xi(t) \rangle = \eta(t) \langle \cdot | \eta(t) \rangle$$

for all $t \in U$.

The map

$$T \longrightarrow \mathbb{R}_+, \quad t \longmapsto \langle \eta(t) | \langle \eta(t_0) | \eta(t) \rangle | \eta(t_0) \rangle = |\langle \eta(t) | \eta(t_0) \rangle|^2$$

is continuous so

$$\lim_{t \rightarrow t_0} |\langle \eta(t_0) | \eta(t) \rangle| = |\langle \eta(t_0) | \eta(t_0) \rangle| \neq 0.$$

Thus U is an open neighborhood of t_0 , ξ is continuous, $\xi(t_0) = \eta(t_0)$, and

$$\xi(t) \langle \cdot | \xi(t) \rangle = \eta(t) \langle \cdot | \eta(t) \rangle$$

for all $t \in U$. ■

Corollary 2.7 *Let u be a positive element of $\mathcal{C}(T, \mathcal{K}(l^2(I)))$.*

- a) *For every $t \in T$ there are an initial segment N_t of \mathbb{N} and an orthonormal family $(\eta_{t,n})_{n \in N_t}$ in $l^2(I)$ such that $\eta_{t,n} = 0$ for all $t \in T \setminus U_n(u)$ and*

$$u(t) = \sum_{n \in N_t} \theta_n(u(t)) \eta_{t,n} \langle \cdot | \eta_{t,n} \rangle.$$

- b) *Let $t_0 \in T$ such that N_{t_0} is finite and let U be a neighborhood of t_0 such that $N_t = N_{t_0}$ for all $t \in U$. Then there is a neighborhood V of t_0 and for every $n \in N_{t_0}$ a continuous map*

$$\xi_n : V \longrightarrow l^2(I)$$

such that for every $t \in V$, $(\xi_n(t))_{n \in N_{t_0}}$ is an orthonormal family in $l^2(I)$ and

$$\xi_n(t) \langle \cdot | \xi_n(t) \rangle = \eta_{t,n} \langle \cdot | \eta_{t,n} \rangle.$$

a) follows from [C] Corollary 6.1.2.13 a \Rightarrow b&c.

b) follows Proposition 2.5 b) and Lemma 2.6. ■

Proposition 2.8 *If u is a positive element of $\mathcal{C}(T, \mathcal{K}(l^2(I)))$ then there is a u -orthonormal sequence $(\xi_n)_{n \in \mathbb{N}}$ in K such that for every*

$$t \in T \setminus \bigcup_{n \in \mathbb{N}} \left(\overline{U_n(u)} \setminus U_n(u) \right),$$

$$u(t) = \sum_{n \in \mathbb{N}} \theta_n(u(t)) (\xi_n(t)) \langle \cdot | \xi_n(t) \rangle \quad (\text{in } \mathcal{K}(l^2(I))).$$

By Corollary 2.7 a), for every $t \in T$ there is an initial segment N_t of \mathbb{N} and an orthonormal family $(\xi_{t,n})_{n \in N_t}$ in $l^2(I)$ such that $\xi_{t,n} = 0$ for all $t \in T \setminus \overline{U_n(u)}$ and $n \in N_t$ and

$$u(t) = \sum_{n \in N_t} \theta_n(u(t)) \xi_{t,n} \langle \cdot | \xi_{t,n} \rangle \quad (\text{in } \mathcal{K}(l^2(I))).$$

For every $k \in \mathbb{N}$, let $f_k \in \mathcal{C}(\mathbb{R}_+)$ with $0 \leq f_k \leq 1$, $f_k = 0$ on $[0, \frac{1}{2k}]$, $f_k = 1$ on $[\frac{1}{k}, \infty]$. By [C] Proposition 4.1.3.20, for every $k \in \mathbb{N}$ the map

$$T \longrightarrow \mathcal{K}(l^2(I)), \quad t \longmapsto f_k(u(t))$$

is continuous. By Proposition 2.1 c), for $t \in T$,

$$f_k(u(t)) = \sum_{n \in N_t} f_k(\theta_n(u(t))) \xi_{t,n} \langle \cdot | \xi_{t,n} \rangle.$$

By Proposition 2.1 a), $(\theta_n(u))_{n \in \mathbb{N}}$ is a decreasing sequence of continuous real functions on T with infimum 0, so by Dini's theorem it converges uniformly to 0 on T . Thus by Proposition 2.1 c), for every $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that

$$\theta_m(f_k(u)) = 0.$$

Since T is hyperstonian and since $U_n(u)$ is the union of a sequence of clopen sets of T (Proposition 2.4 a)), we may assume (by Corollary 2.7 b)) that for every $n \in \mathbb{N}$ there is a $\xi_n \in K$ such that the map

$$U_n(u) \longrightarrow l^2(I), \quad t \longmapsto \xi_n(t)$$

is continuous, with $\langle \xi_n | \xi_n \rangle = e_n(u)$ and $\xi_n(t) \langle \cdot | \xi_n(t) \rangle = \xi_{t,n} \langle \cdot | \xi_{t,n} \rangle$ for all $t \in T$. Moreover for $m, n \in \mathbb{N}$, $m < n$, and $t \in U_n(u)$,

$$\begin{aligned} \xi_m(t) \langle \cdot | \xi_n(t) \rangle \langle \xi_n(t) | \xi_m(t) \rangle &= (\xi_m(t) \langle \cdot | \xi_m(t) \rangle) \circ (\xi_n(t) \langle \cdot | \xi_n(t) \rangle) = \\ &= (\xi_{t,m} \langle \cdot | \xi_{t,m} \rangle) \circ (\xi_{t,n} \langle \cdot | \xi_{t,n} \rangle) = \xi_{t,m} \langle \cdot | \xi_{t,n} \rangle \langle \xi_{t,n} | \xi_{t,m} \rangle = 0. \end{aligned}$$

By Proposition 2.2, $\langle \xi_n(t) | \xi_m(t) \rangle = 0$ so $\langle \xi_n | \xi_m \rangle = 0$. Thus $(\xi_n)_{n \in \mathbb{N}}$ is u -orthonormal. ■

Theorem 2.9 *Let $u \in \mathcal{K}_E(H) (\subset \mathcal{K}_E(K))$.*

a) If u is positive then there is a u -orthonormal sequence $(\xi_n)_{n \in \mathbb{N}}$ in K such that

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \xi_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

In this case $u\xi_n = \theta_n(u)\xi_n \in H$ for all $n \in \mathbb{N}$.

b) There are u -orthonormal sequences $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ in K such that

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

The above identities are called **Schatten decomposition of u** .

By [C] Theorem 5.6.3.5 b), $\mathcal{L}_E(K)$ is a W^* -algebra with \ddot{K} as predual.

a) Let $(\xi_n)_{n \in \mathbb{N}}$ be the u -orthonormal sequence in K defined in Proposition 2.8. By Proposition 2.4 b), for $k, m \in \mathbb{N}$, $k \leq m$,

$$\sum_{n=k}^m \theta_n(u) \xi_n \langle \cdot | \xi_n \rangle \leq \theta_k(u) \sum_{n=k}^m \xi_n \langle \cdot | \xi_n \rangle \leq \theta_k(u),$$

so the sequence $(\theta_n(u) \xi_n \langle \cdot | \xi_n \rangle)_{n \in \mathbb{N}}$ is summable in $\mathcal{K}_E(K)$. By Proposition 2.8 (and [C] Definition 5.6.3.2),

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \xi_n \rangle$$

in $\mathcal{L}_E(K)$ with respect to its weak topology associated to the duality

$$\langle \mathcal{L}_E(K), \ddot{K} \rangle,$$

so

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \xi_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

From $u\xi_n = \theta_n(u)\xi_n$ it follows

$$(u\xi_n)(t) = \theta_n(u(t))\xi_n(t)$$

for all $t \in T$. Thus the map

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle (u\xi_n)(t) | (u\xi_n)(t) \rangle = \theta_n(u(t))^2 \langle \xi_n(t) | \xi_n(t) \rangle$$

is continuous and $u\xi_n \in H$.

b) By a) (and Proposition 2.1 b)), there is a u -orthonormal sequence $(\eta_n)_{n \in \mathbb{N}}$ in K such that

$$|u| = \sum_{n \in \mathbb{N}} \theta_n(u) \eta_n \langle \cdot | \eta_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

Let $u = w|u|$ be the polar representation of u ([C] Theorem 4.4.3.1). Then

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) (w\eta_n) \langle \cdot | \eta_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

For $m, n \in \mathbb{N}$, $m \leq n$, since w^*w is the carrier of $|u|$ and

$$\begin{aligned} |u|\eta_n &= \theta_n(u)\eta_n, \\ \theta_n(u) \langle w\eta_n | w\eta_n \rangle &= \langle \eta_n | w^*w\theta_n(u)\eta_n \rangle = \langle \eta_n | w^*w|u|\eta_n \rangle = \\ &= \langle \eta_n | |u|\eta_n \rangle = \theta_n(u) \langle \eta_n | \eta_n \rangle, \end{aligned}$$

so by Proposition 2.4 b),

$$\langle w\eta_m | w\eta_n \rangle = \delta_{m,n}e_n(u).$$

Thus if we put $\xi_n := w\eta_n$ for every $n \in \mathbb{N}$ then

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

Let $n \in \mathbb{N}$. Since the map

$$U_n(u) \longrightarrow l^2(I), \quad t \longmapsto \eta_n(t)$$

is continuous, the map

$$U_n(u) \longrightarrow l^2(I), \quad t \longmapsto u\eta_n(t)$$

is also continuous. From

$$u\eta_n = \theta_n(u)\xi_n,$$

it follows that the map

$$U_n(u) \longrightarrow l^2(I), \quad t \longmapsto \xi_n(t)$$

is continuous. Thus $(\xi_n)_{n \in \mathbb{N}}$ is a u -orthonormal sequence in K . ■

Proposition 2.10 *Let A be a dense set of T and $(\theta_n)_{n \in \mathbb{N}}$ be a decreasing sequence in E_+ such that*

$$\lim_{n \rightarrow \infty} \theta_n(t) = 0$$

for every $t \in A$. Let further $(\xi_{n,t})_{(n,t) \in \mathbb{N} \times A}$ and $(\eta_{n,t})_{(n,t) \in \mathbb{N} \times A}$ be families in $l^2(I)$ such that $(\xi_{n,t})_{n \in N_t}$ and $(\eta_{n,t})_{n \in N_t}$ are orthonormal families in $l^2(I)$ for all $t \in A$, where

$$N_t := \{ n \in \mathbb{N} \mid \xi_{n,t} \neq 0 \} = \{ n \in \mathbb{N} \mid \eta_{n,t} \neq 0 \}.$$

If for an $u \in \mathcal{K}_E(H)$,

$$\varphi_t u = \sum_{n \in \mathbb{N}} \theta_n(t) \xi_{n,t} \langle \cdot \mid \eta_{n,t} \rangle \quad (\text{in } \mathcal{K}(l^2(I)))$$

for all $t \in A$ then $\theta_n(u) = \theta_n$ for all $n \in \mathbb{N}$.

By [C] Proposition 6.1.2.11, for $t \in A$,

$$(\theta_n(u))(t) = \theta_n(\varphi_t u) = \theta_n(t),$$

so $\theta_n(u) = \theta_n$, since $\theta_n(u)$ is continuous (Proposition 2.1 a)). ■

Corollary 2.11 *Let $u \in \mathcal{K}_E(H)$ and let*

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot \mid \eta_n \rangle$$

be a Schatten decomposition of u .

a)

$$u^* = \sum_{n \in \mathbb{N}} \theta_n(u) \eta_n \langle \cdot \mid \xi_n \rangle$$

is a Schatten decomposition of u^ .*

b) $\theta_n(u^*u) = \theta_n(u)^2$ for every $n \in \mathbb{N}$ and

$$u^*u = \sum_{n \in \mathbb{N}} \theta_n(u)^2 \eta_n \langle \cdot \mid \eta_n \rangle$$

*is a Schatten decomposition of u^*u .*

c) Let N be a subset of \mathbb{N} and

$$v := \sum_{n \in N} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle.$$

If M is an initial segment of \mathbb{N} and $f : M \longrightarrow N$ is an increasing bijective map then

$$\theta_n(v) = \begin{cases} \theta_{f(n)}(u) & \text{if } n \in M \\ 0 & \text{if } n \in \mathbb{N} \setminus M \end{cases}.$$

a) By [C] Proposition 5.6.5.2 a),

$$u^* = \sum_{n \in \mathbb{N}} \theta_n(u) \eta_n \langle \cdot | \xi_n \rangle \quad (\text{in } \mathcal{K}_E(K))$$

and the assertion follows from Proposition 2.1 b).

b) By a), for $n \in \mathbb{N}$,

$$u^* \xi_n = \sum_{m \in \mathbb{N}} \theta_m(u) \eta_m \langle \xi_n | \xi_m \rangle = \theta_n(u) \eta_n,$$

so

$$u^* u = \sum_{n \in \mathbb{N}} \theta_n(u) (u^* \xi_n) \langle \cdot | \eta_n \rangle = \sum_{n \in \mathbb{N}} \theta_n(u)^2 \eta_n \langle \cdot | \eta_n \rangle.$$

If we put

$$\eta'_n : T \longrightarrow l^2(I), \quad t \longmapsto \begin{cases} \eta_n(t) & \text{if } t \in U_n(u) \\ 0 & \text{if } t \in T \setminus U_n(u) \end{cases}$$

for every $n \in \mathbb{N}$ then

$$\varphi_t(u^* u) = \sum_{n \in \mathbb{N}} (\theta_n(u)^2)(t) \eta'_n(t) \langle \cdot | \eta'_n(t) \rangle$$

for all $t \in T$ and the assertion follows from Proposition 2.10.

c) The above defined sequence $(\theta_n(v))_{n \in \mathbb{N}}$ is decreasing and converges to 0. Put

$$A := T \setminus \bigcup_{n \in \mathbb{N}} \left(\overline{U_n(u)} \setminus U_n(u) \right)$$

and for every $n \in \mathbb{N}$ and $t \in A$,

$$\xi_{n,t} := \begin{cases} \xi_{f(n)}(t) & \text{if } n \in M \\ 0 & \text{if } n \in \mathbb{N} \setminus M \end{cases}, \quad \eta_{n,t} := \begin{cases} \eta_{f(n)}(t) & \text{if } n \in M \\ 0 & \text{if } n \in \mathbb{N} \setminus M \end{cases}.$$

Then for $t \in A$,

$$\begin{aligned} \varphi_t(v) &= \sum_{n \in \mathbb{N}} (\theta_n(u))(t) \xi_n(t) \langle \cdot | \eta_n(t) \rangle = \\ &= \sum_{n \in M} (\theta_{f(n)}(u))(t) \xi_{f(n)}(t) \langle \cdot | \eta_{f(n)}(t) \rangle = \sum_{n \in \mathbb{N}} (\theta_n(v))(t) \xi_{n,t} \langle \cdot | \eta_{n,t} \rangle \end{aligned}$$

and the assertion follows from Proposition 2.10. ■

3 The Banach spaces $\mathcal{L}_E^p(H)$

Definition 3.1 We denote for every $p \in [1, \infty[$ by $\mathcal{L}_E^p(\mathbf{H})$ the set of $u \in \mathcal{K}_E(H)$ for which the sequence $(\theta_n^p)_{n \in \mathbb{N}}$ is summable in E and define $\|\cdot\|_p$ by

$$\|\cdot\|_p : \mathcal{L}_E^p(H) \longrightarrow \mathbb{R}_+, \quad u \longmapsto \left\| \sum_{n \in \mathbb{N}} \theta_n(u)^p \right\|^{\frac{1}{p}}.$$

Moreover we put $\mathcal{L}_E^\infty(\mathbf{H}) := \mathcal{L}_E(H)$, $\mathcal{L}_E^0(\mathbf{H}) := \mathcal{K}_E(H)$, and define $\|\cdot\|_0$ by

$$\|\cdot\|_0 : \mathcal{L}_E^0(H) \longrightarrow \mathbb{R}_+, \quad u \longmapsto \|u\| = \|\theta_1(u)\|.$$

Proposition 3.2 Let $u, v \in \mathcal{K}_E(H)$, $0 \leq u \leq v$.

a) $\theta_n(u) \leq \theta_n(v)$ for all $n \in \mathbb{N}$.

b) If $p, q \in [1, \infty[$, $p \leq q$, and $v \in \mathcal{L}_E^p(H)$ then $u \in \mathcal{L}_E^q(H)$.

a) By Corollary 1.4 b), for $t \in T$, $0 \leq \varphi_t u \leq \varphi_t v$ and this implies $\theta_n(\varphi_t u) \leq \theta_n(\varphi_t v)$ ([C] Definition 6.1.2.1).

b) Let $\zeta \in H$. By [C] Theorem 5.6.1.11 c),

$$\langle v\zeta | \zeta \rangle^q = \langle v\zeta | \zeta \rangle^{q-p} \langle v\zeta | \zeta \rangle^p \leq \|v\|^{q-p} \|\zeta\|^{2(q-p)} \langle v\zeta | \zeta \rangle^p,$$

so $\theta_n(v)^q \leq \|v\|^{q-p} \theta_n(v)^p$ for all $n \in \mathbb{N}$ ([C] Definition 6.1.2.1) and therefore $v \in \mathcal{L}_E^q(H)$. By a), $u \in \mathcal{L}_E^q(H)$. ■

Proposition 3.3 *Let $p \in [1, \infty[$.*

a) *If $u \in \mathcal{K}_E(H)_+$ then*

$$u \in \mathcal{L}_E^p(H) \iff u^p \in \mathcal{L}_E^1(H) \implies \|u\|_p^p = \|u^p\|_1.$$

b) *If $u \in \mathcal{K}_E(H)$ then*

$$\begin{aligned} u \in \mathcal{L}_E^p(H) &\iff u^* \in \mathcal{L}_E^p(H) \iff |u| \in \mathcal{L}_E^p(H) \implies \\ &\implies \|u\|_p = \|u^*\|_p = \||u|\|_p. \end{aligned}$$

a) By Proposition 2.1 c), $\theta_n(u^p) = \theta_n(u)^p$ for all $n \in \mathbb{N}$.

b) follows from Proposition 2.1 b). ■

Definition 3.4 *We denote by Ω the set of sequences $(\zeta_n)_{n \in \mathbb{N}}$ in K such that:*

1. *For every $n \in \mathbb{N}$ there is a closed nowhere dense set F_n of T such that the map*

$$T \setminus F_n \longrightarrow l^2(I), \quad t \longmapsto \zeta_n(t)$$

is continuous.

2. *$(\zeta_n(t))_{n \in N_t}$ is an orthonormal family in $l^2(I)$ for all $t \in T$, where*

$$N_t := \{ n \in \mathbb{N} \mid \zeta_n(t) \neq 0 \}.$$

Proposition 3.5 *Let $p \in [1, \infty[$.*

a) If $u \in \mathcal{L}_E^p(H)$ then

$$\sum_{n \in \mathbb{N}} \theta_n(u)^p = \sup \left\{ \sum_{n \in \mathbb{N}} |\langle u \zeta_n | \zeta'_n \rangle|^p \mid (\zeta_n)_{n \in \mathbb{N}}, (\zeta'_n)_{n \in \mathbb{N}} \in \Omega \right\}.$$

b) If u is a positive element of $\mathcal{L}_E^p(H)$ then

$$\sum_{n \in \mathbb{N}} \theta_n(u)^p = \sup \left\{ \sum_{n \in \mathbb{N}} \langle u \zeta_n | \zeta_n \rangle^p \mid (\zeta_n)_{n \in \mathbb{N}} \in \Omega \right\}.$$

a) Let

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

be a Schatten decomposition of u and put for every $n \in \mathbb{N}$

$$\xi'_n : T \longrightarrow l^2(I), \quad t \longmapsto \begin{cases} \xi_n(t) & \text{if } t \in U_n(u) \\ 0 & \text{if } t \in T \setminus U_n(u) \end{cases},$$

$$\eta'_n : T \longrightarrow l^2(I), \quad t \longmapsto \begin{cases} \eta_n(t) & \text{if } t \in U_n(u) \\ 0 & \text{if } t \in T \setminus U_n(u) \end{cases}.$$

Then $(\xi'_n)_{n \in \mathbb{N}}, (\eta'_n)_{n \in \mathbb{N}} \in \Omega$, so

$$\begin{aligned} \sum_{n \in \mathbb{N}} \theta_n(u)^p &= \sum_{n \in \mathbb{N}} |\langle u \eta_n | \xi_n \rangle|^p = \sum_{n \in \mathbb{N}} |\langle u \eta'_n | \xi'_n \rangle|^p \leq \\ &\leq \sup \left\{ \sum_{\lambda \in L} |\langle u \zeta_\lambda | \zeta'_\lambda \rangle|^p \mid (\zeta_\lambda)_{\lambda \in L}, (\zeta'_\lambda)_{\lambda \in L} \in \Omega \right\}. \end{aligned}$$

Let $(\zeta_n)_{n \in \mathbb{N}}, (\zeta'_n)_{n \in \mathbb{N}} \in \Omega$ and $t \in T$. We put for all $m, n \in \mathbb{N}$,

$$\alpha_{m,n} := \langle \xi_n(t) | \zeta'_m(t) \rangle \langle \zeta_m(t) | \eta_n(t) \rangle.$$

If $m \in \mathbb{N}$ then

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\alpha_{m,n}| &= \sum_{n \in \mathbb{N}} |\langle \xi_n(t) | \zeta'_m(t) \rangle \langle \zeta_m(t) | \eta_n(t) \rangle| \leq \\ &\leq \left(\sum_{n \in \mathbb{N}} |\langle \xi_n(t) | \zeta'_m(t) \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{N}} |\langle \zeta_m(t) | \eta_n(t) \rangle|^2 \right)^{\frac{1}{2}} \leq \end{aligned}$$

$$\leq \|\zeta'_m(t)\| \|\zeta_m(t)\| \leq 1.$$

If $n \in \mathbb{N}$ then

$$\begin{aligned} \sum_{m \in \mathbb{N}} |\alpha_{m,n}| &= \sum_{m \in \mathbb{N}} |\langle \xi_n(t) | \zeta'_m(t) \rangle \langle \zeta_m(t) | \eta_n(t) \rangle| \leq \\ &\leq \left(\sum_{m \in \mathbb{N}} |\langle \xi_n(t) | \zeta'_m(t) \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{m \in \mathbb{N}} |\langle \zeta_m(t) | \eta_n(t) \rangle|^2 \right)^{\frac{1}{2}} \leq \\ &\leq \|\xi_n(t)\| \|\eta_n(t)\| \leq 1. \end{aligned}$$

For $m \in \mathbb{N}$,

$$\langle (\varphi_t u) \zeta_m(t) | \zeta'_m(t) \rangle = \sum_{n \in \mathbb{N}} \theta_n(\varphi_t(u)) \langle \xi_n(t) | \zeta'_m(t) \rangle \langle \zeta_m(t) | \eta_n(t) \rangle.$$

By [C] Lemma 6.1.3.9,

$$\sum_{n \in \mathbb{N}} |\langle (\varphi_t u) \zeta_n(t) | \zeta'_n(t) \rangle|^p \leq \sum_{n \in \mathbb{N}} \theta_n(\varphi_t u)^p.$$

Since

$$\langle (\varphi_t u) \zeta_n(t) | \zeta'_n(t) \rangle = (\langle u \zeta_n | \zeta'_n \rangle)(t)$$

for all $t \in T \setminus \bigcup_{n \in \mathbb{N}} F_n$, we get

$$\begin{aligned} \sum_{n \in \mathbb{N}} |\langle u \zeta_n | \zeta'_n \rangle|^p &\leq \sum_{n \in \mathbb{N}} \theta_n(u)^p, \\ \sup \left\{ \sum_{n \in \mathbb{N}} |\langle u \zeta_n | \zeta'_n \rangle|^p \mid (\zeta_n)_{n \in \mathbb{N}}, (\zeta'_n)_{n \in \mathbb{N}} \in \Omega \right\} &\leq \sum_{n \in \mathbb{N}} \theta_n(u)^p. \end{aligned}$$

b) The proof is similar to the proof of a). ■

Theorem 3.6 *Let $p \in [1, \infty[$.*

a) $\mathcal{L}_E^p(H)$ is a vector subspace of $\mathcal{K}_E(H)$ and the map

$$\mathcal{L}_E^p(H) \longrightarrow \mathbb{R}_+, \quad u \longmapsto \|u\|_p$$

is a norm. We always consider $\mathcal{L}_E^p(H)$ endowed with this norm.

b) $\mathcal{L}_E^p(H)$ is complete.

c) If $u \in \mathcal{L}_E^p(H)$ and

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

is a Schatten decomposition of u with $\xi_n, \eta_n \in H$ for all $n \in \mathbb{N}$ then the above sum converges in $\mathcal{L}_E^p(H)$.

a) Let $u, v \in \mathcal{L}_E^p(H)$. By [C] Proposition 6.1.2.5, for $n \in \mathbb{N}$,

$$\theta_{2n-1}(u+v) \leq \theta_n(u) + \theta_n(v),$$

$$\theta_{2n}(u+v) \leq \theta_n(u) + \theta_{n+1}(v),$$

so

$$\theta_{2n-1}(u+v)^p \leq (\theta_n(u) + \theta_n(v))^p \leq 2^{p-1}(\theta_n(u)^p + \theta_n(v)^p),$$

$$\theta_{2n}(u+v)^p \leq (\theta_n(u) + \theta_{n+1}(v))^p \leq 2^{p-1}(\theta_n(u)^p + \theta_{n+1}(v)^p).$$

Thus $u+v \in \mathcal{L}_E^p(H)$. Let $(\xi_n)_{n \in \mathbb{N}}, (\eta_n)_{n \in \mathbb{N}} \in \Omega$. By Proposition 3.5 a),

$$\begin{aligned} \left(\sum_{n \in \mathbb{N}} |\langle (u+v)\xi_n | \eta_n \rangle|^p \right)^{\frac{1}{p}} &= \left(\sum_{n \in \mathbb{N}} |\langle u\xi_n | \eta_n \rangle + \langle v\xi_n | \eta_n \rangle|^p \right)^{\frac{1}{p}} \leq \\ &\leq \left(\sum_{n \in \mathbb{N}} |\langle u\xi_n | \eta_n \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{n \in \mathbb{N}} |\langle v\xi_n | \eta_n \rangle|^p \right)^{\frac{1}{p}} \leq \|u\|_p + \|v\|_p, \\ \|u+v\|_p &\leq \|u\|_p + \|v\|_p. \end{aligned}$$

b) Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}_E^p(H)$. Then $(u_n)_{n \in \mathbb{N}}$ converges in $\mathcal{K}_E(H)$ to a u . Let $\varepsilon > 0$. There is an $n_0 \in \mathbb{N}$ such that

$$\|u_m - u_n\|_p < \varepsilon$$

for all $m, n \in \mathbb{N} \setminus \mathbb{N}_{n_0}$. Let $(\xi_k)_{k \in \mathbb{N}}, (\eta_k)_{k \in \mathbb{N}} \in \Omega$. By a) and Proposition 3.5 a),

$$\left\| \sum_{k \in \mathbb{N}} |\langle (u_m - u_n)\xi_k | \eta_k \rangle|^p \right\| \leq \|u_m - u_n\|_p^p < \varepsilon^p$$

for all $m, n \in \mathbb{N} \setminus \mathbb{N}_{n_0}$. Hence

$$\left\| \sum_{k \in \mathbb{N}} | \langle (u_n - u) \xi_k | \eta_k \rangle |^p \right\| < \varepsilon^p$$

for all $n \in \mathbb{N} \setminus \mathbb{N}_{n_0}$. By a) and Proposition 3.5 a), again,

$$u_n - u \in \mathcal{L}_E^p(H), \quad u \in \mathcal{L}_E^p(H), \quad \|u_n - u\|_p < \varepsilon$$

for all $n \in \mathbb{N} \setminus \mathbb{N}_{n_0}$. Thus $(u_n)_{n \in \mathbb{N}}$ converges to $u \in \mathcal{L}_E^p(H)$ and $\mathcal{L}_E^p(H)$ is complete.

c) By Corollary 2.11 c), for $n_0 \in \mathbb{N}$,

$$\left\| \sum_{n=n_0}^{\infty} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle \right\|_p = \left(\sum_{n=n_0}^{\infty} \theta_n(u)^p \right)^{\frac{1}{p}}. \quad \blacksquare$$

Corollary 3.7 *If $p \in [1, \infty[$, $u \in \mathcal{L}_E^p(H)$, and $v, w \in \mathcal{L}_E(H)$ then*

$$vuw \in \mathcal{L}_E^p(H), \quad \|vuw\|_p \leq \|v\| \|u\|_p \|w\|.$$

By Proposition 1.3 d) and [C] Corollary 6.1.3.13 a), for $t \in T$ and $n \in \mathbb{N}$,

$$\begin{aligned} \theta_n(\varphi_t(vuw)) &= \theta_n((\varphi_t v)(\varphi_t u)(\varphi_t w)) \leq \\ &\leq \|\varphi_t v\| \theta_n(\varphi_t u) \|\varphi_t w\| \leq \|v\| \theta_n(\varphi_t u) \|w\| \end{aligned}$$

and the assertion follows. \blacksquare

Corollary 3.8 *Let $p \in \{0\} \cup [1, \infty[$ and let $q \in [1, \infty]$ be the conjugate exponent of p .*

a) *If $u \in \mathcal{L}_E^p(H)$ and $v \in \mathcal{L}_E^q(H)$ then*

$$uv, vu \in \mathcal{L}_E^1(H),$$

$$\|uv\|_1 \leq \|u\|_p \|v\|_q, \quad \|vu\|_1 \leq \|u\|_p \|v\|_q \quad (\textbf{H\"older inequality}).$$

b) For every $u \in \mathcal{L}_E^p(H)$ there is a $v \in \mathcal{L}_E^q(H)$ such that

$$\|uv\|_1 = \|vu\|_1 = \|u\|_p \|v\|_q.$$

a) By Corollary 3.7 we may assume $p \in]1, \infty[$. By [C] Corollary 6.1.2.7, for $n \in \mathbb{N}$,

$$\theta_{2n-1}(uv) \leq \theta_n(u)\theta_n(v), \quad \theta_{2n}(uv) \leq \theta_n(u)\theta_{n+1}(v),$$

so for $N \subset \mathbb{N}$,

$$\begin{aligned} \sum_{n \in N} \theta_{2n-1}(uv) &\leq \sum_{n \in N} \theta_n(u)\theta_n(v) \leq \left(\sum_{n \in N} \theta_n(u)^p \right)^{\frac{1}{p}} \left(\sum_{n \in N} \theta_n(v)^q \right)^{\frac{1}{q}}, \\ \sum_{n \in N} \theta_{2n}(uv) &\leq \sum_{n \in N} \theta_n(u)\theta_{n+1}(v) \leq \left(\sum_{n \in N} \theta_n(u)^p \right)^{\frac{1}{p}} \left(\sum_{n \in N} \theta_{n+1}(v)^q \right)^{\frac{1}{q}}. \end{aligned}$$

Thus $(\theta_n(uv))_{n \in \mathbb{N}}$ is summable in E and $uv \in \mathcal{L}_E^1(H)$. By [C] Theorem 6.1.3.21, for $t \in T$,

$$\begin{aligned} \sum_{n \in \mathbb{N}} \theta_n(\varphi_t(uv)) &\leq \left(\sum_{n \in \mathbb{N}} \theta_n(\varphi_t(u))^p \right)^{\frac{1}{p}} \left(\sum_{n \in \mathbb{N}} \theta_n(\varphi_t(v))^q \right)^{\frac{1}{q}}, \\ \|uv\|_1 &\leq \|u\|_p \|v\|_q. \end{aligned}$$

The assertion for vu follows.

b) Let

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

be a Schatten decomposition of u . If $p = 1$ then we may take $v = id_H$. Assume $p = 0$. Put

$$v := \eta_1 \langle \cdot | \xi_1 \rangle.$$

By Proposition 2.2, $v \in \mathcal{L}_E^1(H)$, $\|v\|_1 = 1$,

$$\begin{aligned} uv &= \sum_{n \in \mathbb{N}} \theta_n(u) (\xi_n \langle \cdot | \eta_n \rangle)(\eta_1 \langle \cdot | \xi_1 \rangle) = \\ &= \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \eta_1 | \eta_n \rangle \langle \cdot | \xi_1 \rangle = \theta_1(u) \xi_1 \langle \cdot | \xi_1 \rangle, \end{aligned}$$

$$\begin{aligned}
vu &= \sum_{n \in \mathbb{N}} \theta_n(u) (\eta_1 \langle \cdot | \xi_1 \rangle) (\xi_n \langle \cdot | \eta_n \rangle) = \\
&= \sum_{n \in \mathbb{N}} \theta_n(u) \eta_1 \langle \xi_n | \xi_1 \rangle \langle \cdot | \eta_n \rangle = \theta_1(u) \eta_1 \langle \cdot | \eta_1 \rangle.
\end{aligned}$$

Thus (by Proposition 2.2)

$$\|uv\|_1 = \|vu\|_1 = \|\theta_1(u)\| = \|u\|_p \|v\|_q.$$

Assume now $p \in]1, \infty[$. Put

$$v := \sum_{n \in \mathbb{N}} \theta_n(u)^{\frac{p}{q}} \eta_n \langle \cdot | \xi_n \rangle \quad (\text{in } \mathcal{K}_E(K)).$$

By Corollary 2.11 c), $\theta_n(v) = \theta_n(u)^{\frac{p}{q}}$ for every $n \in \mathbb{N}$ so

$$v \in \mathcal{L}_E^q(H), \quad \|v\|_q^q = \|u\|_p^p.$$

For $n \in \mathbb{N}$,

$$u\eta_n = \theta_n(u)\xi_n, \quad v\xi_n = \theta_n(u)^{\frac{p}{q}}\eta_n,$$

so

$$uv = \sum_{n \in \mathbb{N}} \theta_n(u)^{\frac{p}{q}+1} \xi_n \langle \cdot | \xi_n \rangle, \quad vu = \sum_{n \in \mathbb{N}} \theta_n(u)^{1+\frac{p}{q}} \eta_n \langle \cdot | \eta_n \rangle.$$

By Corollary 2.11 c),

$$\begin{aligned}
\theta_n(uv) &= \theta_n(vu) = \theta_n(u)^{\frac{p}{q}+1} = \theta_n(u)^p, \\
\|uv\|_1 &= \|vu\|_1 = \sum_{n \in \mathbb{N}} \theta_n(u)^p = \|u\|_p^p = \\
&= \|u\|_p \|u\|_p^{p-1} = \|u\|_p \|v\|_q^{\frac{q}{p}(p-1)} = \|u\|_p \|v\|_q. \quad \blacksquare
\end{aligned}$$

4 The trace

Proposition 4.1 *Let $(\theta_n)_{n \in \mathbb{N}}$ be a summable sequence in E_+ and let $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ be sequences in $K^\#$.*

a) $(\theta_n \xi_n \langle \cdot | \eta_n \rangle)_{n \in \mathbb{N}}$ is summable in $\mathcal{K}_E(K)$; we put

$$u := \sum_{n \in \mathbb{N}} \theta_n \xi_n \langle \cdot | \eta_n \rangle.$$

b) For every Fourier basis A of K ([C] Definition 5.6.3.11)

$$\sum_{n \in \mathbb{N}} \theta_n \langle \xi_n | \eta_n \rangle = \sum_{\zeta \in A} \langle u \zeta | \zeta \rangle.$$

a) By [C] Proposition 5.6.5.2 a),

$$\|\xi_n \langle \cdot | \eta_n \rangle\| \leq \|\xi_n\| \|\eta_n\| \leq 1$$

for every $n \in \mathbb{N}$.

b) For $\zeta \in A$,

$$\langle u \zeta | \zeta \rangle = \sum_{n \in \mathbb{N}} \theta_n \langle \xi_n | \zeta \rangle \langle \zeta | \eta_n \rangle.$$

By [C] Theorem 5.6.3.13 f), since the above sum converges uniformly,

$$\begin{aligned} \sum_{\zeta \in A} \langle u \zeta | \zeta \rangle &= \sum_{\zeta \in A} \sum_{n \in \mathbb{N}} \theta_n \langle \xi_n | \zeta \rangle \langle \zeta | \eta_n \rangle = \\ &= \sum_{n \in \mathbb{N}} \theta_n \sum_{\zeta \in A} \langle \xi_n | \zeta \rangle \langle \zeta | \eta_n \rangle = \sum_{n \in \mathbb{N}} \theta_n \langle \xi_n | \eta_n \rangle. \end{aligned} \quad \blacksquare$$

Definition 4.2 Let $u \in \mathcal{L}_E^1(H)$ and let

$$u := \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

be a Schatten decomposition of u . We put

$$\mathbf{tr} \, u := \sum_{n \in \mathbb{N}} \theta_n(u) \langle \xi_n | \eta_n \rangle \in E$$

and call it **the trace of u** (by Proposition 4.1 b) the trace of u does not depend on the chosen Schatten decomposition of u).

Corollary 4.3 Given $u \in \mathcal{L}_E(K)$ and $\xi, \xi', \eta, \eta' \in K$,

$$\begin{aligned} \text{tr}(\xi \langle \cdot | \eta \rangle) &= \langle \xi | \eta \rangle, \\ \text{tr}(u \circ (\xi \langle \cdot | \eta \rangle)) &= \langle u\xi | \eta \rangle = \text{tr}((\xi \langle \cdot | \eta \rangle) \circ u), \\ \text{tr}((\xi \langle \cdot | \eta \rangle) \circ (\xi' \langle \cdot | \eta' \rangle)) &= \langle \xi | \eta' \rangle \langle \xi' | \eta \rangle. \end{aligned}$$

[C] Proposition 5.6.5.2 d), e). ■

Proposition 4.4 We put for all $u \in \mathcal{L}_E(H)$ and $x \in E$,

$$ux : H \longrightarrow H, \quad \xi \longmapsto (u\xi)x = u(\xi x).$$

Then $ux \in \mathcal{L}_E(H)$, $(ux)^* = u^*x^*$, and $\|ux\| \leq \|u\| \|x\|$ for all $u \in \mathcal{L}_E(H)$ and $x \in E$,

For $\xi, \eta \in H$,

$$\begin{aligned} \langle (ux)\xi | \eta \rangle &= \langle (u\xi)x | \eta \rangle = \langle u\xi | \eta \rangle x = \\ &= \langle \xi | u^*\eta \rangle x = \langle \xi | (u^*\eta)x^* \rangle = \langle \xi | (u^*x^*)\eta \rangle, \end{aligned}$$

so $ux \in \mathcal{L}_E(H)$ and $(ux)^* = u^*x^*$. For $\xi \in H$,

$$\|(ux)\xi\| = \|(u\xi)x\| \leq \|u\xi\| \|x\| \leq \|u\| \|\xi\| \|x\|,$$

so $\|ux\| \leq \|u\| \|x\|$. ■

Corollary 4.5 The map

$$\mathcal{L}_E^1(H) \longrightarrow E, \quad u \longmapsto \text{tr } u$$

is linear, involutive, positive, and continuous with norm 1 (Theorem 3.6 a)) and

$$\|\text{tr } u\| = \|u\|_1$$

for every positive element of $\mathcal{L}_E^1(H)$. Moreover for all $u \in \mathcal{L}_E^1(H)$ and $x \in E$ (Proposition 4.4),

$$\text{tr}(ux) = (\text{tr } u)x.$$

tr is linear (Proposition 4.1 b)), involutive (Corollary 2.11 a)), and continuous with norm at most 1 ([C] proposition 5.6.5.2 a)). By Definition 4.2, tr is positive and

$$\|\text{tr } u\| = \|u\|_1$$

If A is a Fourier basis of K then by Proposition 4.1 b) ,

$$\text{tr}(ux) = \sum_{\zeta \in A} \langle (ux)\zeta \mid \zeta \rangle = \left(\sum_{\zeta \in A} \langle u\zeta \mid \zeta \rangle \right) x = (\text{tr } u)x. \quad \blacksquare$$

Corollary 4.6 *If $u \in \mathcal{K}_E(H)_+$ and $p \in [1, \infty[$ then*

$$u \in \mathcal{L}_E^p(H) \iff u^p \in \mathcal{L}_E^1(H) \implies \|u\|_p = (\text{tr } u^p)^{\frac{1}{p}}.$$

By Proposition 3.3 a), $u \in \mathcal{L}_E^p(H)$ iff $u^p \in \mathcal{L}_E^1(H)$ and

$$\|u\|_p^p = \|u^p\|_1.$$

By Corollary 4.5,

$$\|u\|_p = (\text{tr } u^p)^{\frac{1}{p}}. \quad \blacksquare$$

Proposition 4.7 *If $u \in \mathcal{L}_E^1(H)$ and $v \in \mathcal{L}_E(H)$ then (Corollary 3.7)*

$$\text{tr}(uv) = \text{tr}(vu).$$

Let

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot \mid \eta_n \rangle$$

be a Schatten decomposition of u . By [C] Proposition 5.6.5.2 d),e) (and [C] Theorem 5.6.4.7 d)),

$$\begin{aligned} \text{tr}(vu) &= \text{tr} \sum_{n \in \mathbb{N}} \theta_n(u) (v\xi_n) \langle \cdot \mid \eta_n \rangle = \sum_{n \in \mathbb{N}} \theta_n(u) \langle v\xi_n \mid \eta_n \rangle = \\ &= \sum_{n \in \mathbb{N}} \theta_n(u) \langle \xi_n \mid v^* \eta_n \rangle = \text{tr} \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot \mid v^* \eta_n \rangle = \text{tr}(uv). \quad \blacksquare \end{aligned}$$

5 Hilbert-Schmidt operators

Definition 5.1 *The elements of $\mathcal{L}_E^2(H)$ are called **Hilbert-Schmidt operators on H** .*

Proposition 5.2 $\mathcal{L}_E^2(H)$ *endowed with the exterior multiplications (Corollary 3.7)*

$$\begin{aligned}\mathcal{L}_E(H) \times \mathcal{L}_E^2(H) &\longrightarrow \mathcal{L}_E^2(H), & (w, u) &\longmapsto wu, \\ \mathcal{L}_E^2(H) \times \mathcal{L}_E(H) &\longrightarrow \mathcal{L}_E^2(H), & (u, w) &\longmapsto uw\end{aligned}$$

and with the inner-product (Corollary 3.8 a))

$$\langle \cdot | \cdot \rangle : \mathcal{L}_E^2(H) \times \mathcal{L}_E^2(H) \longrightarrow \mathcal{L}_E(H), \quad (u, v) \longmapsto v^*u$$

is a unital Hilbert $\mathcal{L}_E(H)$ -module ([C] Definition 5.6.1.4).

For $u, v \in \mathcal{L}_E^2(H)$ and $w \in \mathcal{L}_E(H)$,

$$\begin{aligned}\langle u | v \rangle^* &= (v^*u)^* = u^*v = \langle v | u \rangle, \\ \langle uw | v \rangle &= v^*(uw) = (v^*u)w = \langle u | v \rangle w, \\ \langle wu | v \rangle &= v^*(wu) = (w^*v)^*u = \langle u | w^*v \rangle, \\ \langle wu | wu \rangle &= u^*w^*wu \leq \|w\|^2 u^*u = \|w\|^2 \langle u | u \rangle, \\ 1_{\mathcal{L}_E(H)}u &= u.\end{aligned}$$

Moreover if $\mathbb{K} = \mathbb{R}$,

$$(\langle u | u \rangle + \langle v | v \rangle, \langle v | u \rangle - \langle u | v \rangle) = (u^*u + v^*v, u^*v - v^*u) = (u, v)^*(u, v)$$

is a positive element of the complexification of $\mathcal{L}_E(H)$. ■

Proposition 5.3 *For every $u \in \mathcal{K}_E(H)$,*

$$u \in \mathcal{L}_E^2(H) \iff u^*u \in \mathcal{L}_E^1(H) \implies \|u^*u\|_1 = \|u\|_2^2.$$

If $u \in \mathcal{L}_E^2(H)$ then by Corollary 2.11 b), $u^*u \in \mathcal{L}_E^1(H)$ and

$$\|u^*u\|_1 = \sum_{n \in \mathbb{N}} \theta_n(u^*u) = \sum_{n \in \mathbb{N}} \theta_n(u)^2 = \|u\|_2^2.$$

If $u^*u \in \mathcal{L}_E^1(H)$ then by Corollary 2.11 b), $(\theta_n(u)^2)_{n \in \mathbb{N}}$ is summable in E so $u \in \mathcal{L}_E^2(H)$. ■

Theorem 5.4

a) $u, v \in \mathcal{L}_E^2(H) \implies v^*u \in \mathcal{L}_E^1(H)$.

b) $\mathcal{L}_E^2(H)$ endowed with the exterior multiplication (Proposition 4.4)

$$\mathcal{L}_E^2(H) \times E \longrightarrow \mathcal{L}_E^2(H), \quad (u, x) \longmapsto ux$$

and with the inner-product (a))

$$\langle \cdot | \cdot \rangle : \mathcal{L}_E^2(H) \times \mathcal{L}_E^2(H) \longrightarrow E, \quad (u, v) \longmapsto \text{tr}(v^*u)$$

is a Hilbert right E -module with norm $\|\cdot\|_2$.

c) $u, v \in \mathcal{L}_E^2(H) \implies \langle u | v \rangle = \langle v^* | u^* \rangle$.

a) follows from the Hölder inequality.

b) For $u, v \in \mathcal{L}_E^2(H)$ and $x \in E$, by Corollary 4.5 and Proposition 5.3,

$$\langle ux | v \rangle = \text{tr}(v^*ux) = \text{tr}(v^*u)x = \langle u | v \rangle x,$$

$$\langle u | v \rangle = \text{tr}(v^*u) = (\text{tr}(u^*v))^* = \langle v | u \rangle^*,$$

$$\langle u | u \rangle = \text{tr}(u^*u) \in E_+, \quad \|\langle u | u \rangle\| = \|u\|_2^2.$$

c) By Proposition 4.7,

$$\langle u | v \rangle = \text{tr}(v^*u) = \text{tr}(uv^*) = \langle v^* | u^* \rangle. \quad \blacksquare$$

6 Duals of $\mathcal{L}_E^p(H)$ -spaces

Proposition 6.1 *Let $p \in [1, \infty[$ and let \mathcal{F} be the set of $u \in \mathcal{L}_E^p(H)$ for which there is a Schatten decomposition*

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

such that $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ are sequences in H . Then \mathcal{F} is dense in $\mathcal{L}_E^p(H)$.

Let $u \in \mathcal{L}_E^p(H)$ and let

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

be a Schatten decomposition of u . We put for all $n, k \in \mathbb{N}$,

$$U_{n,k} := \left\{ t \in T \left| \theta_n(t) > \frac{1}{kn^2} \right. \right\},$$

$$e_{n,k} : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} 1 & \text{if } t \in \overline{U_{n,k}} \\ 0 & \text{if } t \in T \setminus \overline{U_{n,k}} \end{cases},$$

$$u_k := \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n e_{n,k} \rangle = \sum_{n \in \mathbb{N}} (\theta_n(u) e_{n,k}) (\xi_n e_{n,k}) \langle \cdot | \eta_n e_{n,k} \rangle.$$

For $k \in \mathbb{N}$,

$$\begin{aligned} u - u_k &= \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n (1_E - e_{n,k}) \rangle = \\ &= \sum_{n \in \mathbb{N}} (\theta_n(u) (1_E - e_{n,k})) (\xi_n (1_E - e_{n,k})) \langle \cdot | \eta_n (1_E - e_{n,k}) \rangle. \end{aligned}$$

By Proposition 2.10, for $n, k \in \mathbb{N}$,

$$\theta_n(u - u_k) = \theta_n(u) (1_E - e_{n,k}) \leq \frac{1}{kn^2},$$

so $(\theta_n(u - u_k)^p)_{n \in \mathbb{N}}$ is summable in E and

$$\sum_{n \in \mathbb{N}} \theta_n(u - u_k)^p \leq \frac{1}{k^p} \sum_{n \in \mathbb{N}} \frac{1}{n^{2p}}.$$

Thus $(u_k)_{k \in \mathbb{N}}$ converges to u in $\mathcal{L}_E^p(H)$ and this proves the assertion since $u_k \in \mathcal{F}$ for every $k \in \mathbb{N}$. ■

Theorem 6.2 *Let $p \in \{0\} \cup [1, \infty[$, $q \in [1, \infty]$ the conjugate exponent of p , and $\mathcal{L}(\mathcal{L}_E^p(H), E)$ the involutive Banach space of operators from $\mathcal{L}_E^p(H)$ to E ([C] Proposition 2.3.2.22 a)), the involution being defined for every $\phi \in \mathcal{L}(\mathcal{L}_E^p(H), E)$ by*

$$\phi^* : \mathcal{L}_E^p(H) \longrightarrow E, \quad u \longmapsto (\phi(u^*))^*.$$

Further let \mathcal{G} be the set of $\phi \in \mathcal{L}(\mathcal{L}_E^p(H), E)$ such that

$$1. \quad u \in \mathcal{L}_E^p(H), \quad x \in E \implies \phi(ux) = \phi(u)x$$

$$2. \quad \text{For } \xi \in H,$$

$$(\phi(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I}, (\phi^*(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I} \in H,$$

where for every $\iota \in I$,

$$e_\iota := (\delta_{\iota, \lambda} 1_E)_{\lambda \in I} \in H.$$

a) \mathcal{G} is an involutive vector subspace of $\mathcal{L}(\mathcal{L}_E^p(H), E)$.

b) If we put for every $v \in \mathcal{L}_E^q(H)$ (by the Hölder inequality and Proposition 4.7)

$$\tilde{v} : \mathcal{L}_E^p(H) \longrightarrow E, \quad u \longmapsto \text{tr}(uv) = \text{tr}(vu)$$

then $\tilde{v} \in \mathcal{G}$ and the map

$$\Psi : \mathcal{L}_E^q(H) \longrightarrow \mathcal{G}, \quad v \longmapsto \tilde{v}$$

is an isomorphism of involutive Banach spaces.

a) is easy to see.

b) For $u \in \mathcal{L}_E^p(H)$, by Corollary 4.5 and the Hölder inequality,

$$\|\tilde{v}(u)\| = \|\text{tr}(uv)\| \leq \|uv\|_1 \leq \|u\|_p \|v\|_q,$$

so $\|\tilde{v}\| \leq \|v\|_q$ and $\tilde{v} \in \mathcal{L}(\mathcal{L}_E^p(H), E)$. By Corollary 4.5, for $u \in \mathcal{L}_E^p(H)$ and $x \in E$,

$$\tilde{v}(ux) = \text{tr}(vux) = \text{tr}(vu)x = \tilde{v}(u)x.$$

For $\xi \in H$, by Corollary 4.3,

$$(\tilde{v}(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I} = \text{tr}(v(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I} = (\langle v\xi | e_\iota \rangle)_{\iota \in I} = v\xi \in H,$$

$$(\tilde{v}^*(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I} = v^*\xi \in H,$$

so $\tilde{v} \in \mathcal{G}$. Ψ is obviously linear. For $u \in \mathcal{L}_E^p(H)$, by Corollary 4.5,

$$\tilde{v}^*(u) = \text{tr}(uv^*) = (\text{tr}(vu^*))^* = (\tilde{v}(u^*))^* = \tilde{v}^*(u),$$

so $\tilde{v}^* = \tilde{v}^*$ and Ψ is involutive. Moreover by Corollary 3.8, Ψ is norm preserving. The only thing we have still to prove is the surjectivity of Ψ .

Let $\phi \in \mathcal{G}$ and put ([C] Proposition 5.6.5.2 a))

$$v : H \longrightarrow H, \quad \xi \longmapsto (\phi(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I},$$

$$w : H \longrightarrow H, \quad \xi \longmapsto (\phi^*(\xi \langle \cdot | e_\iota \rangle))_{\iota \in I}.$$

For $\xi, \eta \in H$, by 1. and [C] Proposition 5.6.5.2 a),c),

$$\langle v\xi | \eta \rangle = \sum_{\iota \in I} \langle v\xi | e_\iota \rangle \eta_\iota^* = \sum_{\iota \in I} \phi(\xi \langle \cdot | e_\iota \rangle) \eta_\iota^* = \phi(\xi \langle \cdot | \eta \rangle),$$

$$\|v\xi\|^2 = \|\langle v\xi | v\xi \rangle\| = \|\phi(\xi \langle \cdot | v\xi \rangle)\| \leq \|\phi\| \|\xi\| \|v\xi\|,$$

$$\|v\xi\| \leq \|\phi\| \|\xi\|, \quad \|v\| \leq \|\phi\|.$$

For $\iota, \lambda \in I$, by [C] Proposition 5.6.5.2 a),

$$\begin{aligned} \langle ve_\lambda | e_\iota \rangle &= \phi(e_\lambda \langle \cdot | e_\iota \rangle) = \phi(e_\lambda \langle \cdot | e_\iota \rangle)^{**} = \\ &= (\phi^*(e_\iota \langle \cdot | e_\lambda \rangle))^* = \langle we_\iota | e_\lambda \rangle^* = \langle e_\lambda | we_\iota \rangle. \end{aligned}$$

Thus $v \in \mathcal{L}_E(H)$ and $v^* = w$. Let $u \in \mathcal{L}_E^p(H)$ and let

$$u = \sum_{n \in \mathbb{N}} \theta_n(u) \xi_n \langle \cdot | \eta_n \rangle$$

be a Schatten decomposition of u with $(\xi_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$ sequences in H . Then by the above and Theorem 3.6 c),

$$\tilde{v}(u) = \sum_{n \in \mathbb{N}} \theta_n(u) \tilde{v}(\xi_n \langle \cdot | \eta_n \rangle) = \sum_{n \in \mathbb{N}} \theta_n(u) \phi(\xi_n \langle \cdot | \eta_n \rangle) = \phi(u).$$

By Proposition 6.1, $\tilde{v} = \phi$ and Ψ is surjective. ■

7 Integral operators

Throughout this section S is a compact space, μ a positive Radon measure on S , $(h_\iota)_{\iota \in I}$ an orthonormal basis of $L^2(\mu)$, $H := \bigoplus_{\iota \in I} E$, and $w \in \mathcal{C}(S \times S, E)$. Moreover \odot denotes the algebraic tensor product

Proposition 7.1 *The linear map*

$$L^2(\mu) \odot E \longrightarrow H, \quad f \otimes x \longmapsto (\langle f | h_\iota \rangle x)_{\iota \in I}$$

can be extended to an isomorphism $L^2(\mu) \otimes E \longrightarrow H$ ([L] pages 34-35) of Hilbert right modules.

We denote by Φ the above map. For $(f, x), (g, y) \in L^2(\mu) \times E$ and $z \in E$,

$$\begin{aligned} \langle \Phi(f \otimes x) | \Phi(g \otimes y) \rangle &= \langle (\langle f | h_\iota \rangle x)_{\iota \in I} | (\langle g | h_\iota \rangle y)_{\iota \in I} \rangle = \\ &= \sum_{\iota \in I} y^* \langle h_\iota | g \rangle \langle f | h_\iota \rangle x = y^* \langle f | g \rangle x = \langle f \otimes x | g \otimes y \rangle, \\ \Phi((f \otimes x)z) &= \Phi(f \otimes (xz)) = (\langle f | h_\iota \rangle (xz))_{\iota \in I} = \\ &= (\langle f | h_\iota \rangle x)_{\iota \in I} z = (\Phi(f \otimes x))z, \end{aligned}$$

i.e. Φ preserves the inner-product and the right multiplication so it can be extended to a linear map

$$\Psi : L^2(\mu) \otimes E \longrightarrow H$$

preserving the inner-product and the right multiplication. Moreover

$$\Psi(h_\lambda \otimes z) = (\delta_{\lambda, \iota} z)_{\iota \in I}$$

for all $\lambda \in I$ and $z \in E$, so Ψ is surjective. ■

Lemma 7.2 *The vector subspace of $\mathcal{C}(S \times S, E)$ generated by maps of the form*

$$S \times S \longrightarrow E, \quad (r, s) \longmapsto u(r)v(s),$$

where $u \in \mathcal{C}(S, E)$ and $v \in \mathcal{C}(S, \mathbb{K})$ is dense in $\mathcal{C}(S \times S, E)$.

Let $\varepsilon > 0$. There are finite open coverings $(U_j)_{j \in J}, (V_k)_{k \in K}$ of S such that

$$\|w(r, s) - w(r', s')\| < \varepsilon$$

for all $(j, k) \in J \times K$ and $(r, s), (r', s') \in U_j \times V_k$. Take $r_j \in U_j$ and $s_k \in V_k$ for all $j \in J$ and $k \in K$ and let $(f_j)_{j \in J}$ and $(g_k)_{k \in K}$ be partitions of unity subordinate to the coverings $(U_j)_{j \in J}$ and $(V_k)_{k \in K}$ of S , respectively. For $r, s \in S$,

$$\begin{aligned} & \left\| w(r, s) - \sum_{(j,k) \in J \times K} f_j(r) g_k(s) w(r_j, s_k) \right\| = \\ & = \left\| \sum_{(j,k) \in J \times K} f_j(r) g_k(s) (w(r, s) - w(r_j, s_k)) \right\| \leq \\ & \leq \sum_{(j,k) \in J \times K} f_j(r) g_k(s) \varepsilon = \varepsilon. \end{aligned}$$

If we put

$$u_k : S \longrightarrow E, \quad r \longmapsto \sum_{j \in J} f_j(r) w(r_j, s_k)$$

and $v_k := g_k$ for all $k \in K$ then for $r, s \in S$,

$$\begin{aligned} \sum_{(j,k) \in J \times K} f_j(r) g_k(s) w(r_j, s_k) &= \sum_{k \in K} \left(\sum_{j \in J} f_j(r) w(r_j, s_k) \right) g_k(s) = \\ &= \sum_{k \in K} u_k(r) v_k(s). \end{aligned} \quad \blacksquare$$

Definition 7.3 A function $f : S \times T \longrightarrow \mathbb{K}$ is called **E- μ -integrable** if $f(s, \cdot) \in E$ and $f(\cdot, t) \in \mathcal{L}^1(\mu)$ for all $(s, t) \in S \times T$ and if the map

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \int f(\cdot, t) d\mu$$

is continuous, i.e. it belongs to E . We denote this element of E by

$$\int g d\mu = \int g(s) d\mu(s),$$

where

$$g : S \longrightarrow E, \quad s \longmapsto f(s, \cdot).$$

Lemma 7.4 For every $f \in L^2(\mu)$ the map

$$\tilde{f} : S \longrightarrow E, \quad r \longmapsto \int w(r, s) f(s) \, d\mu(s)$$

is continuous.

Let $r_0 \in S$ and $\varepsilon > 0$. There is a neighborhood U of r_0 such that

$$\sup_{s \in S} \|w(r, s) - w(r_0, s)\| < \varepsilon$$

for all $r \in U$. Then for $r \in U$,

$$\|\tilde{f}(r) - \tilde{f}(r_0)\| = \left\| \int (w(r, s) - w(r_0, s)) f(s) \, d\mu(s) \right\| \leq \varepsilon \int |f(s)| \, d\mu(s). \quad \blacksquare$$

Lemma 7.5 We use the notation of Lemma 7.4.

a) The linear map

$$L^2(\mu) \odot E \longrightarrow \mathcal{C}(S, E), \quad f \odot x \longmapsto \tilde{f}x$$

is continuous so it can be extended by continuity to an operator

$$L^2(\mu) \otimes E \longrightarrow \mathcal{C}(S, E).$$

b) The linear map

$$L^2(\mu) \odot E \longrightarrow H, \quad f \odot x \longmapsto \tilde{f}x$$

is continuous so it can be extended by continuity to an operator

$$\tilde{w} : H \longrightarrow H.$$

a) Let $(f_j)_{j \in J}$ and $(x_j)_{j \in J}$ be finite families in $L^2(\mu)$ and E , respectively. For $r \in S$,

$$\left| \left(\sum_{j \in J} \tilde{f}_j x_j \right) (r) \right| = \left| \sum_{j \in J} \int w(r, s) f_j(s) x_j \, d\mu(s) \right| =$$

$$\begin{aligned}
&= \left| \int w(r, s) \left(\sum_{j \in J} f_j(s) x_j \, d\mu(s) \right) \right| \leq \\
&\int |w(r, s)| \left| \sum_{j \in J} f_j(s) x_j \right| d\mu(s) \leq \|w\| \int \left| \sum_{j \in J} f_j(s) x_j \right| d\mu(s),
\end{aligned}$$

where

$$\|w\| := \sup_{r, s \in S} \|w(r, s)\|.$$

Thus

$$\begin{aligned}
\left| \left(\sum_{j \in J} \tilde{f}_j x_j \right) (r) \right| &\leq \|w\| \mu(S)^{\frac{1}{2}} \left(\int \left| \sum_{j \in J} f_j(s) x_j \right|^2 d\mu(s) \right)^{\frac{1}{2}} = \\
&= \|w\| \mu(S)^{\frac{1}{2}} \left(\sum_{j, k \in J} x_j x_k^* \int f_j(s) \overline{f_k(s)} d\mu(s) \right)^{\frac{1}{2}} = \\
&= \|w\| \mu(S)^{\frac{1}{2}} \left(\sum_{j, k \in J} \langle f_j | f_k \rangle \langle x_j | x_k \rangle \right)^{\frac{1}{2}} = \\
&= \|w\| \mu(S)^{\frac{1}{2}} \left\langle \sum_{j \in J} (f_j \otimes x_j) \left| \sum_{j \in J} (f_j \otimes x_j) \right. \right\rangle^{\frac{1}{2}} \leq \\
&\leq \|w\| \mu(S)^{\frac{1}{2}} \left\| \sum_{j \in J} (f_j \otimes x_j) \right\|.
\end{aligned}$$

b) By [W] T3.13,

$$\mathcal{C}(S, E) \approx \mathcal{C}(S, \mathbb{K}) \otimes E$$

and by Proposition 7.1, $L^2(\mu) \otimes E \approx H$. The assertion follows from the continuity of the inclusion $\mathcal{C}(S, \mathbb{K}) \otimes E \subset L^2(\mu) \otimes E$. ■

Theorem 7.6 *We use the notation of Lemma 7.5 b). $\tilde{w} \in \mathcal{L}_E^2(H)$ (i.e. \tilde{w} is a Hilbert Schmitt operator on H) and $\tilde{w}^* = \tilde{w}'$, where*

$$w' : S \times S \longrightarrow E, \quad (r, s) \longmapsto w(s, r)^*$$

and \tilde{w}' is defined similarly to \tilde{w} .

Step 1 $\tilde{w} \in \mathcal{L}_E(H)$ and $\tilde{w}^* = \tilde{w}'$

For $(f, x), (g, y) \in L^2(\mu) \times E$,

$$\begin{aligned}
\langle \tilde{w}(f \otimes x) | g \otimes y \rangle &= \int y^* g(r)^* \left(\int w(r, s) f(s) x \, d\mu(s) \right) d\mu(r) = \\
&= \int f(s) x \left(\int w(r, s) y^* g(r)^* d\mu(r) \right) d\mu(s) = \\
&= \int f(s) x \left(\int w(r, s)^* g(r) y \, d\mu(r) \right)^* d\mu(s) = \\
&= \int f(s) x \left(\tilde{w}'(g \otimes y) \right)^* (s) \, d\mu(s) = \langle f \otimes x | \tilde{w}'(g \otimes y) \rangle
\end{aligned}$$

so $\tilde{w} \in \mathcal{L}_E(H)$ and $\tilde{w}^* = \tilde{w}'$.

Step 2 $\tilde{w} \in \mathcal{K}_E(H)$

By Lemma 7.2, we may assume that there are $u \in \mathcal{C}(S, E)$ and $v \in \mathcal{C}(S, \mathbb{K})$ with

$$w : S \times S \longrightarrow E, \quad (r, s) \longmapsto u(r)v(s).$$

For $(f, x) \in L^2(\mu) \times E$,

$$\begin{aligned}
\tilde{w}(f \otimes x) &= \int u v(s) f(s) x \, d\mu(s) = u \langle f | \bar{v} \rangle \langle x | 1_E \rangle = \\
&= u \langle f \otimes x | \bar{v} \otimes 1_E \rangle = (u \langle \cdot | \bar{v} \otimes 1_E \rangle)(f \otimes x), \\
\tilde{w} &= u \langle \cdot | \bar{v} \otimes 1_E \rangle \in \mathcal{K}_E(H).
\end{aligned}$$

Step 3 $\tilde{w} \in \mathcal{L}_E^2(H)$

For $t \in T$,

$$(w(\cdot, \cdot))(t) \in \mathcal{C}(S \times S, \mathbb{K}) \subset L^2(\mu \otimes \mu),$$

so we consider in the sequel $(w(\cdot, \cdot))(t) \in L^2(\mu \otimes \mu)$.

Let $t_0 \in T$ and $\varepsilon > 0$. There is a neighborhood U of t_0 such that

$$\sup_{r,s \in S} |(w(r,s))(t) - (w(r,s))(t_0)| < \varepsilon$$

for all $t \in U$. Then

$$\begin{aligned} & \| (w(\cdot, \cdot))(t) - (w(\cdot, \cdot))(t_0) \|_2^2 = \\ &= \int |(w(r,s))(t) - (w(r,s))(t_0)|^2 d(\mu \otimes \mu)(r,s) \leq \varepsilon^2 \mu(S)^2 \end{aligned}$$

for all $t \in U$. Thus the map

$$T \longrightarrow L^2(\mu \otimes \mu), \quad t \longmapsto (w(\cdot, \cdot))(t)$$

is continuous. By [C] Proposition 6.1.4.9 a), the map

$$L^2(\mu \otimes \mu) \longrightarrow \mathcal{L}^2(L^2(\mu)), \quad k \longmapsto \widehat{k}$$

is an isometry of Banach spaces. Since for all $t \in T$

$$\varphi_t \tilde{w} = \widehat{(w(\cdot, \cdot))(t)}$$

we get

$$(\theta_n(\tilde{w}))(t) = \theta_n(\varphi_t \tilde{w}) = \theta_n(\widehat{(w(\cdot, \cdot))(t)})$$

for all $n \in \mathbb{N}$ and so

$$\sum_{n \in \mathbb{N}} (\theta_n(\tilde{w}))(t)^2 = \sum_{n \in \mathbb{N}} \theta_n(\widehat{(w(\cdot, \cdot))(t)})^2 = \|(w(\cdot, \cdot))(t)\|_2^2.$$

Thus the map

$$T \longrightarrow \mathbb{R}, \quad t \longmapsto \sum_{n \in \mathbb{N}} (\theta_n(\tilde{w}))(t)^2$$

is continuous and $\tilde{w} \in \mathcal{L}_E^2(H)$. ■

REFERENCES

- [C] Constantinescu, Corneliu, C*-algebras. Elsevir, 2001.
- [L] Lance, E. Christopher, Hilbert C*-modules, Cambridge University Press, 1995.
- [W] Wegge-Olsen, N. E., K-Theory and C*-Algebras, Oxford University Press, 1993.

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